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FINITE ELEMENT ANALYSIS OF PRIMARY AND SECONDARY CONSOLIDATION

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Abstract—In this paper a finite element formulation for a two phase soil with a viscoelastic skeleton is developed. A method of integration is proposed which considerably reduces the body of hereditary information necessary to carry forward the solution and it is shown that under certain circumstances this integration method is unconditionally stable.

INTRODUCTION

Saturated soil is a two phase material consisting of a solid skeleton and water filled voids. When an increment of stress is suddenly applied to an element of saturated soil there is an instantaneous increase in pore pressure and excess pore pressures develop. For most soils the pore water is incompressible compared to the soil skeleton and so initially the soil element deforms at constant volume. After this initial deformation the pore fluid has a tendency to move from areas of higher excess pore pressure to areas of lower excess pore pressure and the element undergoes additional deformation and the soil is said to consolidate.

The process of consolidation under one dimensional conditions was first investigated by Terzaghi[1] and was subsequently extended to three dimensional conditions by Biot[2]. These authors assumed that the flow of the pore water was governed by Darcy's law and more important that the response of the soil skeleton was elastic. The above assumption lead to the conclusion that a soil element subjected to a change in stress will undergo an initial deformation and then consolidate to a final deformed state after all excess pore pressures are dissipated. For many soils this is not the case and the soil displays a tendency to creep under constant load even though all excess pore pressures have been dissipated. The presence of creep or secondary consolidation can be explained by supposing that the soil skeleton is a viscoelastic solid. The general equations governing the behaviour of such a material were developed by Biot[3].

It is extremely difficult to find analytic solutions to the equations of viscoelastic consolidating soil and the only solutions which have been found are those to problems having the simplest geometry and involving the simplest boundary conditions[4, 5]. This is perhaps not surprising since such problems combine the difficulty of viscoelastic stress analysis coupled with the complexity of a diffusion process. It is clear that to solve more complicated problems it is necessary to develop numerical methods.

The purpose of this paper is to develop a general technique for the numerical solution of the equations of viscoelastic consolidation. In order to do this the basic equations are formulated in terms of Laplace transforms, these equations are then approximated by the finite element technique. The approximate equations are then inverted and a forward marching solution technique is developed. It is shown that under certain conditions this numerical process is unconditionally stable.

BASIC EQUATIONS

For the sake of definiteness consider a saturated soil with a viscoelastic skeleton occupying a region V with a surface S. Portion of the surface S_T is subject to specified tractions **T** while the remainder of the surface S_D is subject to zero displacement. It is also assumed the portion of the surface S_p is free to drain while the remainder S_I is impermeable.†

†It is not difficult to incorporate more complicated boundary conditions, both elastic and hydraulic, into the theory. The procedure is straight forward and will not be given here.

In the following section it will be convenient to adopt the following notation:

x_i are the components of the position vector $\mathbf{x}$ relative to a fixed rectangular Cartesian reference system
u_i are the components of the displacement vector $\mathbf{u}$
$\epsilon_{ij} = 1/2[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)]$ are the components of the strain tensor
σ_{ij} denote the increase in stress components, tension reckoned positive, due to the applied tractions
p is the excess pore pressure due to the applied tractions
$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}$ are the components of the effective stress tensor
n_i the components of the outward normal to the surface S.

It is assumed that the process is a quasistatic one and thus the stresses must be in equilibrium so that

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{1a}$$

The relationship between effective stress and strain can be written in the form

$$\sigma'_{ij} = Q_{ijkl}\{\epsilon_{kl}\} \tag{1b}$$

where Q_{ijkl} is a viscoelastic operator defined by the relation

$$Q_{ijkl}\{\epsilon_{kl}\} = R_{ijkl}(t)\epsilon_{kl}(0) + \int_0^t R_{ijkl}(t-\tau)\dot{\epsilon}_{kl}(\tau)\,\mathrm{d}\tau. \tag{2b}$$

and $R_{ijkl}(t)$ are the relaxation functions for the soil skeleton.

The flow of the pore water is governed by a generalised Darcy's law

$$v_i = -\frac{k_{ij}}{\gamma_w}\frac{\partial p}{\partial x_j}. \tag{1c}$$

where v_i are the components of the superficial velocity vector of the pore water, k_{ij} are the components of the permeability tensor, and γ_w is the unit weight of water.

Now as was mentioned in the introduction, for many soils the pore water is relatively incompressible compared to the soil and it therefore follows that the volume change of a soil element will equal the volume of water squeezed out and hence

$$\frac{\partial \theta}{\partial t} = -\frac{\partial v_i}{\partial x_i}. \tag{1d}$$

where $\theta = \epsilon_{ii}$ is the volume strain of the soil.

The field eqns (1a, b, c, d) must be solved subject to the boundary conditions

$$\sigma_{ij}n_j = T_i \quad \text{on} \quad S_T \tag{2a}$$

$$u_i = 0 \quad \text{on} \quad S_D \tag{2b}$$

$$p = 0 \quad \text{on} \quad S_p \tag{2c}$$

$$n_j v_j = 0 \quad \text{on} \quad S_I \tag{2d}$$

and the initial condition.

$$\theta = 0 \quad \text{when} \quad t = 0+. \tag{3}$$

This last equation follows from the observation that when the loads are first applied the pore water can only escape from the element at a finite rate and thus the element cannot undergo an instantaneous volume change.

Equations (1)–(2) are simplified considerably when expressed in terms of Laplace transforms

$$\bar{f} = \int_0^\infty e^{-st} f(t)\, dt$$

and may be written in the form

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} = 0 \tag{4a}$$

$$\bar{\sigma}'_{ij} = \bar{A}_{ijkl}\bar{\epsilon}_{kl} \tag{4b}$$

$$\bar{v}_i = \frac{-k_{ij}}{\gamma_w}\frac{\partial \bar{p}}{\partial x_j} \tag{4c}$$

$$s\bar{\theta} = \frac{-\partial \bar{v}_i}{\partial x_i} \tag{4d}$$

where

$$\bar{A}_{ijkl} = s\bar{R}_{ijkl}$$

$$\bar{\sigma}_{ij}n_j = \bar{T}_i \quad \text{on} \quad S_T \tag{5a}$$

$$\bar{u}_i = 0 \quad \text{on} \quad S_D \tag{5b}$$

$$\bar{p} = 0 \quad \text{on} \quad S_p \tag{5c}$$

$$\frac{\partial \bar{p}}{\partial x_j} n_j = 0 \quad \text{on} \quad S_I. \tag{5d}$$

These eqns (4) and (5) are analogous to those developed for a material with an elastic skeleton[6].

FINITE ELEMENT FORMULATION

Finite element formulations for an elastic consolidating soil have been proposed by several authors[7–13]. Many of these authors have adopted a Gurtin variational formulation, however in this paper the authors will extend the approach developed in Ref. [13] and obtain the finite element approximating equations in terms of the Laplace transforms of the field quantities.

In developing a finite element formulation for the problem of a viscoelastic consolidating soil it is convenient to rewrite several of the equations developed in the previous section in the following alternative notation: $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{31}, \sigma_{12}, \sigma_{23})^T$ is the vector of stress components. $\boldsymbol{\epsilon} = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{31}, 2\epsilon_{12}, 2\epsilon_{23})^T$ is the vector of strain components. $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\mathbf{e}$ is the vector of effective stress components where $\mathbf{e} = (1, 1, 1, 0, 0, 0)^T$.

Equation (1b) becomes

$$\boldsymbol{\sigma}' = \mathbf{Q}\{\boldsymbol{\epsilon}\} \tag{6a}$$

where $\mathbf{Q}$ is a viscoelastic operator which can be written in the form

$$\mathbf{Q}\{\boldsymbol{\epsilon}\} = \mathbf{R}(t)\boldsymbol{\epsilon}(0) + \int_0^t \mathbf{R}(t-\tau)\dot{\boldsymbol{\epsilon}}(\tau)\, d\tau \tag{6b}$$

where $\mathbf{R}$ is the relaxation matrix of the soil skeleton.

When eqns (6a, b) are written in terms of Laplace transforms they become

$$\bar{\boldsymbol{\sigma}}' = \bar{\mathbf{D}}\bar{\boldsymbol{\epsilon}} \tag{6c}$$

where $\bar{\mathbf{D}} = s\bar{\mathbf{R}}$ is the matrix of "transformed" elastic constants. Similarly Darcy's law can be written in the form

$$\mathbf{v} = -\frac{1}{\gamma_w}\mathbf{k}\nabla p \tag{6d}$$

where $\mathbf{k}$ is the matrix of permeability coefficients.

In developing a finite element formulation it is convenient to recast eqns (4) and (5) in the form of a variational principle. It can be shown by a slight extension of the methods developed in Ref. [6, 13] that the solution of eqns (4) and (5) is equivalent to finding a displacement field $\bar{\mathbf{u}}$ and a pore pressure field $\bar{p}$ which satisfy the boundary conditions (5b, c) and render the functional Ψ stationary:

$$\Psi(\bar{\mathbf{u}}, \bar{p}) = \int_V \left\{\frac{1}{2}\bar{\boldsymbol{\epsilon}}^T\bar{\mathbf{D}}\bar{\boldsymbol{\epsilon}} - \bar{p}\bar{\theta} - \frac{1}{2\gamma_w s}\nabla\bar{p}^T\mathbf{k}\nabla\bar{p}\right\}\mathrm{d}V - \int_{S_T} \bar{\mathbf{u}}^T\bar{\mathbf{T}}\,\mathrm{d}S. \tag{7}$$

The minimization problem described in eqn (7) can be solved approximately by the finite element technique as follows:

(i) Suppose that the displacement field $\mathbf{u}$ and excess pore pressure field p can be adequately represented by their values at nodes $1, 2, \ldots$

$$\boldsymbol{\delta}^T = (\mathbf{u}_1^T, \mathbf{u}_2^T, \ldots)$$

$$\mathbf{q} = (p_1, p_2, \ldots)$$

The subscripts in the above definitions refer to values at a particular node.

(ii) Suppose that the continuous fields $\mathbf{u}$, p can be adequately approximated in terms of their nodal values.†

$$\mathbf{u} = \mathbf{C}(\mathbf{x})\boldsymbol{\delta}$$

$$p = \mathbf{a}^T(\mathbf{x})\mathbf{q}.$$

(iii) The strains, volume strain and pore pressure gradients may then be written in terms of nodal values

$$\boldsymbol{\epsilon} = \mathbf{B}(\mathbf{x})\boldsymbol{\delta}$$

$$\theta = \mathbf{d}^T(\mathbf{x})\boldsymbol{\delta}$$

$$\nabla p = \mathbf{E}(\mathbf{x})\mathbf{q}$$

where

$$\mathbf{B} = \partial[\mathbf{C}]$$

$$\mathbf{E} = \begin{bmatrix} \partial\mathbf{a}^T/\partial x_1 \\ \partial\mathbf{a}^T/\partial x_2 \\ \partial\mathbf{a}^T/\partial x_3 \end{bmatrix}$$

$$\mathbf{d}^T = \mathbf{e}^T\mathbf{B}$$

and

$$\partial = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_3 & \partial/\partial x_2 \end{bmatrix}$$

†$\mathbf{C}(\mathbf{x})$, $\mathbf{a}(\mathbf{x})$ are both known; their precise form will depend upon the particular finite element adopted.

(iv) The functional Ψ can now be approximated by the quadratic form

$$\Psi_{approx} = \frac{1}{2}\left\{\bar{\boldsymbol{\delta}}^T\bar{\mathbf{K}}\bar{\boldsymbol{\delta}} - 2\bar{\boldsymbol{\delta}}^T\mathbf{L}^T\bar{\mathbf{q}} - \frac{1}{s}\bar{\mathbf{q}}^T\boldsymbol{\Phi}\bar{\mathbf{q}} - 2\bar{\boldsymbol{\delta}}^T\bar{\mathbf{f}}\right\}$$

where

$$\bar{\mathbf{K}} = \int_V \mathbf{B}^T\bar{\mathbf{D}}\mathbf{B}\,\mathrm{d}V$$

is the "transformed" stiffness matrix.

$$\mathbf{L} = \int_V \mathbf{a}\mathbf{d}^T\,\mathrm{d}V$$

$$\boldsymbol{\Phi} = \int_V \frac{1}{\gamma_w}\mathbf{E}^T\mathbf{k}\mathbf{E}\,\mathrm{d}V$$

$$\bar{\mathbf{f}} = \int_{S_T} \mathbf{C}^T\bar{\mathbf{T}}\,\mathrm{d}S$$

is the "transformed" vector of nodal forces. Minimisation of Ψ_{approx} leads to the set of approximating equations

$$\begin{bmatrix} \bar{\mathbf{K}}, & -\mathbf{L}^T \\ -\mathbf{L}, & -(1/s)\boldsymbol{\Phi} \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{\delta}} \\ \bar{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{f}} \\ \mathbf{0} \end{bmatrix}. \tag{8a}$$

The Laplace transform may now be inverted to yield

$$\mathrm{S}\{\boldsymbol{\delta}\} - \mathbf{L}^T\mathbf{q} = \mathbf{f} \tag{8b}$$

$$-\mathbf{L}\boldsymbol{\delta} - \boldsymbol{\Phi}\int_0^t \mathbf{q}\,\mathrm{d}t = \mathbf{0}$$

where S is a viscoelastic stiffness operator defined as follows

$$\mathrm{S}\{\boldsymbol{\delta}\} = \mathbf{Z}(t)\boldsymbol{\delta}(0) + \int_0^t \mathbf{Z}(t-\tau)\dot{\boldsymbol{\delta}}(\tau)\,\mathrm{d}\tau$$

or

$$\mathrm{S}\{\boldsymbol{\delta}\} = \mathbf{Z}(0)\boldsymbol{\delta}(t) + \int_0^t \dot{\mathbf{Z}}(t-\tau)\boldsymbol{\delta}(\tau)\,\mathrm{d}\tau$$

where

$$\mathbf{Z}(t) = \int_V \mathbf{B}^T\mathbf{R}(t)\mathbf{B}\,\mathrm{d}V$$

NUMERICAL SOLUTION

Equation (8b) can be regarded as a set of Volterra integral equations, the numerical solution of such equations is in principle quite simple, see for example Ref. [14].

Thus for example, if integrals were approximated by a trapezoidal rule, an approximation to eqn (8b) would be

$$\begin{bmatrix} 1/2(\mathbf{Z}(\Delta t) + \mathbf{Z}(0)), & -\mathbf{L}^T \\ -\mathbf{L}, & -(\Delta t/2)\boldsymbol{\Phi} \end{bmatrix}\begin{bmatrix} \boldsymbol{\delta}(t) \\ \mathbf{q}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{a}(t) \\ \mathbf{b}(t) \end{bmatrix} \tag{9}$$

where

$$\mathbf{a}(t) = \mathbf{f}(t) - \frac{1}{2}(\mathbf{Z}(0) - \mathbf{Z}(\Delta t))\boldsymbol{\delta}(t-\Delta t) - \int_0^{t-\Delta t} \mathbf{Z}(t-\tau)\dot{\boldsymbol{\delta}}(\tau)\,\mathrm{d}\tau$$

$$\mathbf{b}(t) = -\mathbf{L}\boldsymbol{\delta}(t-\Delta t) + \frac{\Delta t}{2}\boldsymbol{\Phi}\mathbf{q}(t-\Delta t).$$

Clearly the right hand side or load vector of eqn (9) depends only on values of $\boldsymbol{\delta}(\tau)$, $\mathbf{q}(\tau) 0 \leq \tau \leq t-\Delta t$ and thus the solution can be found by a "forward marching" procedure.

While the above approach is conceptually simple it has the disadvantage that in order to calculate the load vector it is necessary to know the values of $\boldsymbol{\delta}, \mathbf{q}$ at all previous values of time. This involves considerable computation and a great deal of storage thus making the solution of any but the simplest problem an extremely time consuming process.

One method of overcoming this disadvantage is to adopt the approach of Christian and Watt[18] and represent the soil response as a truncated Prony series

$$\mathbf{R}(t)=\mathbf{D}_0+\mathbf{D}_1 \mathrm{e}^{-\gamma_1 t}+\cdots+\mathbf{D}_n \mathrm{e}^{-\gamma_n t}$$

thus the "matrix of transformed" elastic constants has the form

$$\bar{\mathbf{D}}(s)=\mathbf{D}_0+\frac{s}{s+\gamma_1} \mathbf{D}_1+\cdots+\frac{s}{s+\gamma_n} \mathbf{D}_n. \tag{10b}$$

In the discussion of stability of the numerical method it will prove necessary to assume that the matrices D_0, $\mathbf{D}_1, \ldots, \mathbf{D}_n$ are positive definite. This is certainly true for an isotropic material whose volumetric and deviatoric response can be represented in terms of mechanical models consisting of springs and dashpots connected in series and parallel[15, 16]. When $\bar{\mathbf{D}}$ has the form (10b) the matrix $\bar{\mathbf{K}}$ can also be written in the form

$$\bar{\mathbf{K}}=\mathbf{K}_0+\sum_{i=1}^{n} \frac{s}{s+\gamma_i} \mathbf{K}_i. \tag{11a}$$

Now in order to facilitate numerical analysis introduce the auxilary quantities

$$\bar{\boldsymbol{\rho}}_i=\frac{s}{s+\gamma_i} \bar{\boldsymbol{\delta}} \tag{11b}$$

so that

$$\bar{\boldsymbol{\delta}}=\left(1+\frac{\gamma_i}{s}\right) \bar{\boldsymbol{\rho}}_i. \tag{11c}$$

The quantities $\boldsymbol{\delta}$, $\boldsymbol{\rho}$ are therefore related by

$$\boldsymbol{\rho}_i(t)=\boldsymbol{\delta}(t)-\gamma_i \int_0^t \mathrm{e}^{-\gamma_i(t-\tau)} \boldsymbol{\delta}(\tau)\, \mathrm{d} \tau \tag{11d}$$

or

$$\boldsymbol{\delta}(t)=\boldsymbol{\rho}_i(t)+\gamma_i \int_0^t \boldsymbol{\rho}(\tau)\, \mathrm{d} \tau. \tag{11e}$$

Equation (11e) may now be approximated as

$$\boldsymbol{\delta}_{k+1}-\boldsymbol{\delta}_k=\boldsymbol{\rho}_{i,k+1}-\boldsymbol{\rho}_{ik}+\gamma_i \Delta t(\beta \boldsymbol{\rho}_{i,k+1}+(1-\beta) \boldsymbol{\rho}_{i,k}) \tag{12a}$$

$$\boldsymbol{\rho}_{i,k+1}-\boldsymbol{\rho}_{i,k}=\frac{\boldsymbol{\delta}_{k+1}-\boldsymbol{\delta}_k-\gamma_i \Delta t \boldsymbol{\rho}_{i,k}}{1+\beta \gamma_i \Delta t} \tag{12b}$$

where the subscript k indicates the value of a quantity at time $t_k=(k-1)\Delta t$. Different values of the parameter correspond to different integration rules, for example $\beta=1/2$ corresponds to trapezoidal integration.

Equation (8a) can be written in terms of the variables ρ_i in the form

$$\mathbf{K}_0\bar{\boldsymbol{\delta}} + \sum_{i=1}^{n} \mathbf{K}_i\bar{\boldsymbol{\rho}}_i - \mathbf{L}^T\bar{\mathbf{q}} = \bar{\mathbf{f}} \tag{13a}$$

$$-\mathbf{L}\bar{\boldsymbol{\delta}} - \frac{1}{s}\boldsymbol{\Phi}\bar{\mathbf{q}} = \mathbf{0}.$$

Now inverting the Laplace transform and making use of eqn (12b) it is found that

$$\begin{bmatrix} \mathbf{K} - \mathbf{L}^T \\ -\mathbf{L} - \Delta t\beta\boldsymbol{\Phi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_{k+1} - \boldsymbol{\delta}_k \\ \mathbf{q}_{k+1} - \mathbf{q}_k \end{bmatrix} = \begin{bmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{bmatrix} \tag{13b}$$

where

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{n} \frac{\mathbf{K}_i}{1 + \beta\gamma_i\Delta t}$$

$$\mathbf{a}_k = \sum_{i=1}^{n} \frac{\gamma_i\Delta t}{1 + \beta\gamma_i\Delta t}\,\mathbf{K}_i\boldsymbol{\rho}_{i,k} + \mathbf{f}_{k+1} - \mathbf{f}_k$$

$$\mathbf{b}_k = -\Delta t\boldsymbol{\Phi}\mathbf{q}_k.$$

Equation (12) can be used to find the values of $\boldsymbol{\delta}_{k+1}$, $\mathbf{q}_{k+1}$ from the known values of $\boldsymbol{\delta}_k$, $\mathbf{q}_k$, $\boldsymbol{\rho}_{i,k}$. Once these values have been found $\boldsymbol{\rho}_{i,k+1}$ can be found from eqn (12b) and so the process may be repeated and the calculation marched forward.

It is apparent that the difficulties associated with eqn (9) have been largely overcome. It is no longer necessary to store the values of $\boldsymbol{\delta}$ for all previous times but merely to store the current values of $\mathbf{q}$, $\boldsymbol{\delta}$, $\boldsymbol{\rho}_i$, in essence all the hereditary information necessary for the calculation to proceed is contained in $\boldsymbol{\rho}_i$.

It is shown in the Appendix that this integration scheme is unconditionally stable provided $\beta \geq 1/2$ and stable when $0 \leq \beta \leq 1/2$ provided

$$\Delta t < \frac{\operatorname{Min} \lambda_i}{(1/2 - \beta)} \tag{14}$$

where λ_i are the eigenvalues of a certain matrix.

Equation (14) can be used as a stability criterion once the smallest eigenvalue λ_i is known. In most circumstances it is more convenient to circumvent this calculation by selecting $\beta \geq 1/2$.

EXAMPLES

As a first illustration of the theory consider the problem of a clay layer resting on a rigid impermeable base and consolidating under one dimensional conditions as shown in Fig. 1a. An analytic solution to this problem has been found by Gibson[4] and Christie[17].

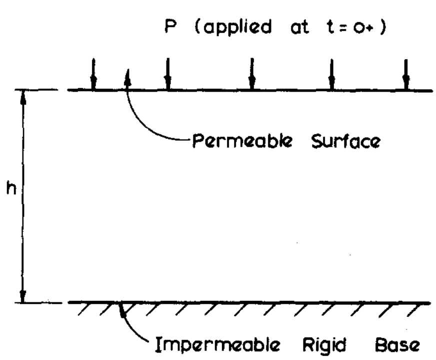

Fig. 1(a).

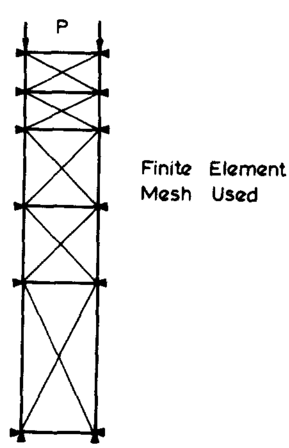

Fig. 1(b).

A numerical solution to the problem was found using the finite element network shown in Fig. 1b. for a variety of viscoelastic models. For this and other problems examined in this section triangular elements with a linear variation in displacement and pore pressure were used although there is no difficulty associated with the use of other elements.

First it was assumed that the soil had a constant Poisson's ratio ν_0 and that its response in shear could be obtained from the two parameter model shown in Fig. 2. If such a material is

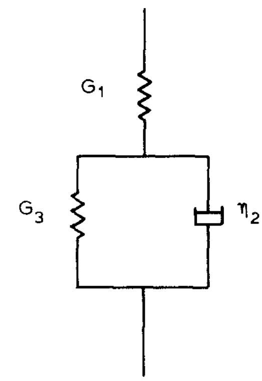

Fig. 2.

subjected to a constant shear stress τ_0 and there are no consolidation effects, it will sustain a shear strain γ given by

$$\gamma = \tau_0\left\{\frac{1}{G_0}e^{-T_s} + \frac{1}{G_\infty}(1 - e^{-T_s})\right\} \tag{15}$$

where

$$G_0 = G_1 = \text{shear "molulus" at } t = 0$$

$$G_\infty = \frac{G_1G_3}{G_1 + G_3} = \text{shear "modulus" at } t = \infty$$

$$T_s = G_3 t/\eta$$

and will thus creep from an initial strain $\gamma_0 = \tau_0/G_0$ to a final strain $\gamma_\infty = \tau_0/G_\infty$.

In presenting the results it is useful to adopt the following notation due to Gibson:

$$M = G_0/G_\infty = \text{the compressibility factor.} \tag{16a}$$

$$T_v = \frac{(2-2\nu_0)G_0kt}{(1-2\nu_0)\gamma_w h^2} = \text{the primary time factor} \tag{16b}$$

$$N = \frac{2-2\nu_0}{1-2\nu_0}\frac{T_s}{T_v} = \text{the time influence factor relating the consolidation and viscoelastic time scales.} \tag{16c}$$

The degree of settlement calculated from the numerical solution and that calculated from the analytic solution are shown in Fig. 3 for the particular case of Poisson's ratio zero, a compressibility factor $M = 3.33$ and a variety of time influence factors N varying, from $N = \infty$ where consolidation occurs much more slowly than creep, to $N=0$ where consolidation occurs much more quickly than creep.

To show the effect of a different choice of rheological model the problem described above was solved for the case in which the shear behaviour varied according to eqn (15) but the bulk modulus was constant. For the case of comparison it was assumed that the two materials had the same instantaneous response and thus:

$$\bar{K} = K_0 = \frac{2G_0(1+\nu_0)}{3(1-2\nu_0)}. \tag{17}$$

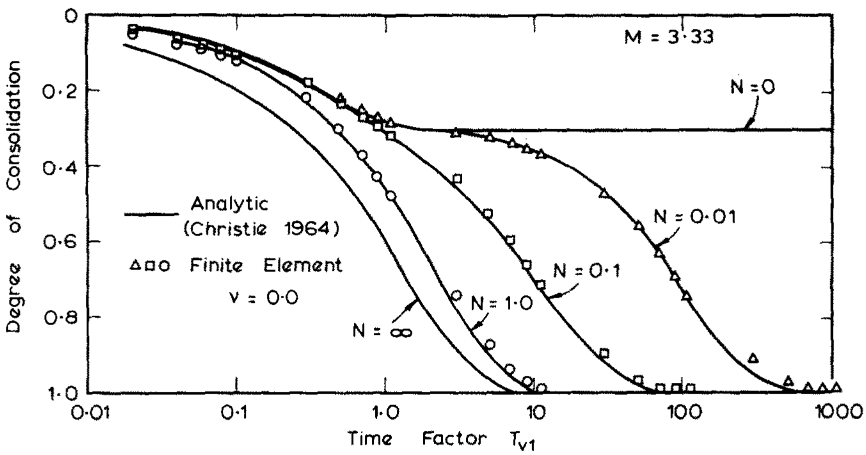

Fig. 3. Plot of degree of consolidation vs time factor for a viscoelastic material under one dimensional conditions.

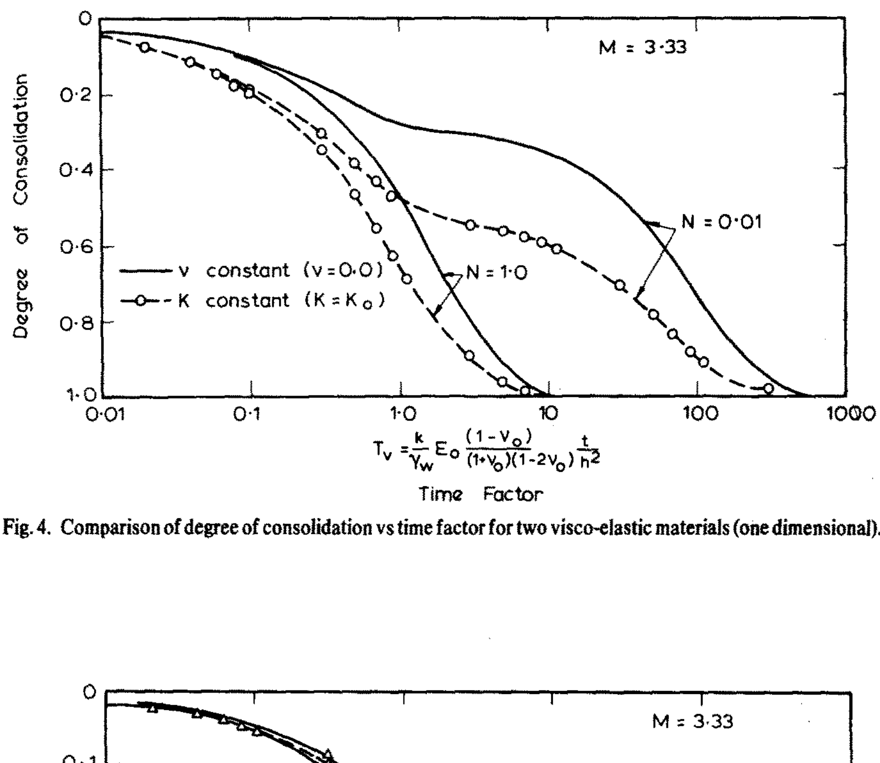

Fig. 4. Comparison of degree of consolidation vs time factor for two visco-elastic materials (one dimensional).

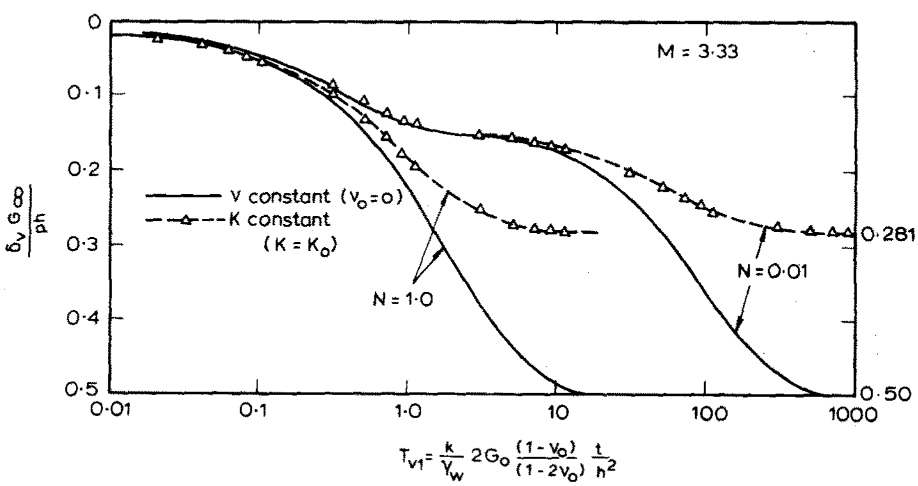

Fig. 5. Comparison of vertical surface deflection vs time factor for two visco-elastic materials (one dimensional).

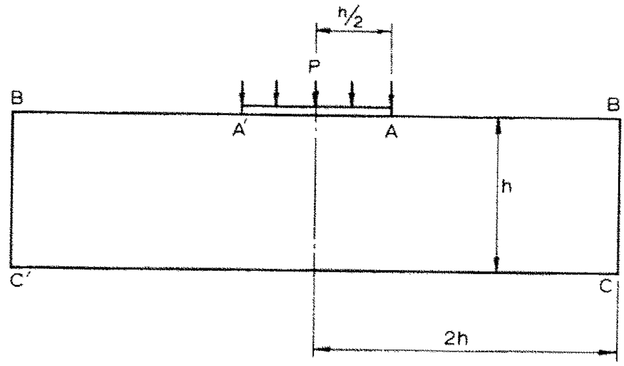

Fig. 6(a).

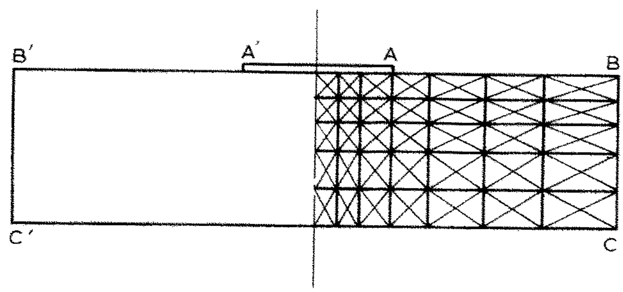

Fig. 6(b).

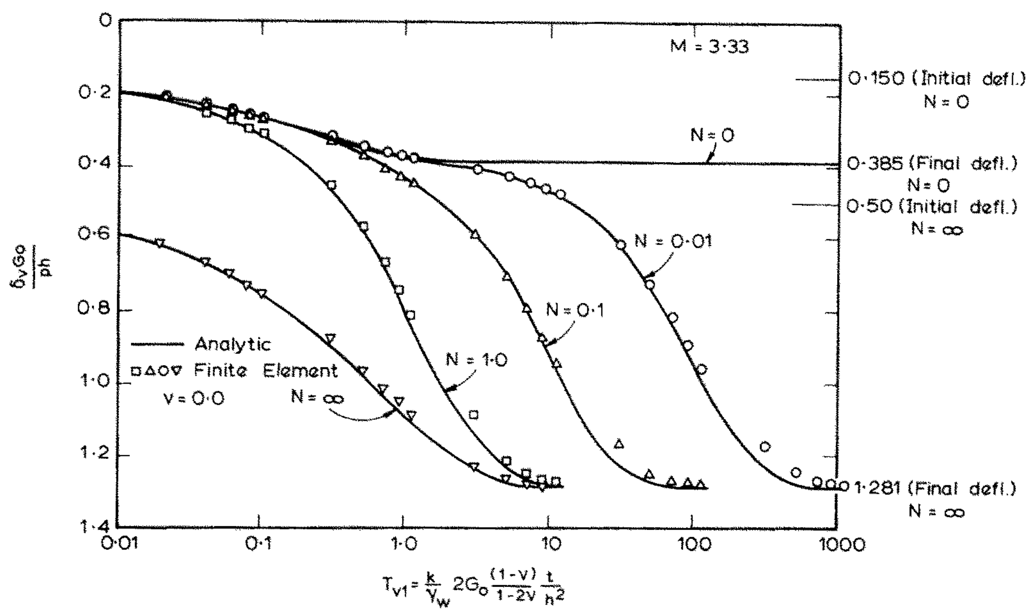

Fig. 7. Settlement vs time factor plot for the central point of a strip footing on a visco-elastic layer (surface laterally restrained).

The degree of settlement and the actual settlement for the two different viscoelastic behaviours are shown in Figs. 4 and 5 and it can be seen that although the solutions behave similarly for early time they are substantially different for the moderate and large times.

The second example taken was a two dimensional problem shown schematically in (Fig. 6a). The problem is that of a flexible strip footing AA' on a layer of visco-elastic soil $ABCC'B'A'$. The base $C'C$ is smooth, rigid and impermeable, the sides $C'B'$ and BC are shear free, while the surface $B'A'AB$ is permeable and restrained against any horizontal movements. This particular problem is not a practical one but was chosen because it is possible to find an analytic solution using a method similar to that described by Freudenthal[5].

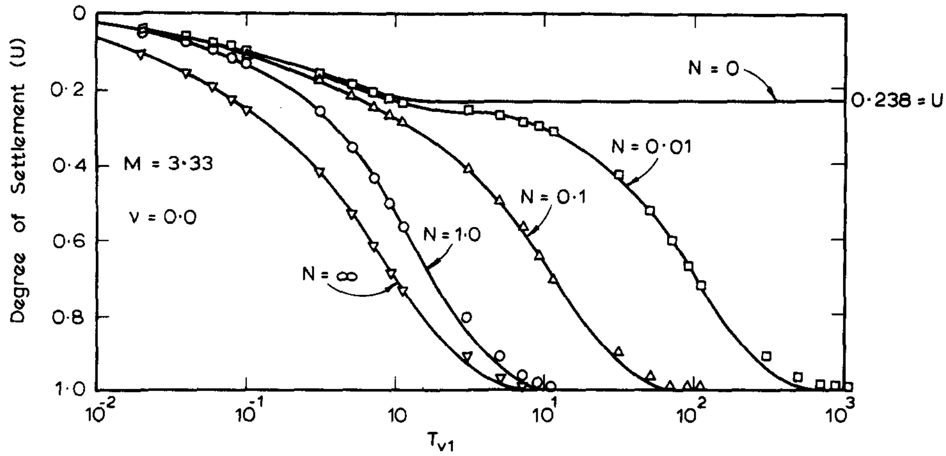

Fig. 8. Plot of degree of consolidation vs time factor for central point of strip footing on a visco-elastic material (rough based layer).

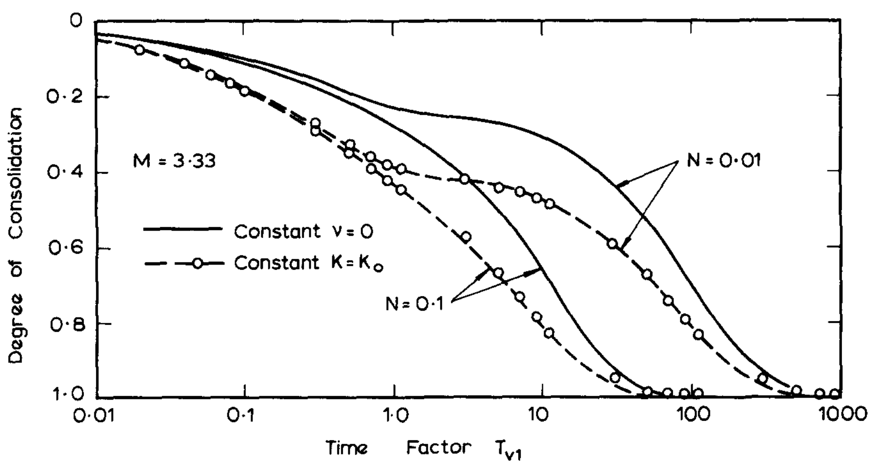

Fig. 9. Comparison of degree of consolidation vs time factor for two visco-elastic materials (strip footing on rough based layer).

The finite element mesh used for the problem is shown in (Fig. 6b). Results of the analysis are shown in (Fig. 7) for the case where $M = 3.33$ and Poisson's ratio is zero. In this figure the numerical and analytic values of the central vertical deflection are compared for various values of N. Again there is good agreement between analytic and finite element solutions.

Finally a more realistic two-dimensional problem was tried. The problem can again be shown schematically by (Fig. 6a). AA' is the flexible footing sitting on the layer $ABCC'B'A'$. The base CC' is rough, rigid, and impermeable, while the sides $C'B'$, BC are shear free. Drainage occurs across the surface $B'A'AB$. The finite element mesh used is the same as that in (Fig. 6b).

Figure 8 shows the degree of settlement plotted against time factor for various values of N for the case where Poisson's ratio is zero.

The effect of assuming a different viscoelastic behaviour and supposing that the bulk modulus is constant and defined by eqn (17) may be seen in (Fig. 9) and it is again found that there is a significant change in the degree of settlement.

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APPENDIX

In order that the integration scheme proposed in this paper, may be used with any confidence it is necessary to establish the conditions under which it remains stable. The problem of the stability of numerical integration schemes for an elastic consolidating soil has been investigated and illustrated by Booker and Small[13] and it is found that similar criteria hold for viscoelastic consolidation.

The stability of the integration scheme described by eqns (13) will depend upon the behaviour of the solution of the homogeneous equations

$$\mathbf{K}_0\boldsymbol{\delta}_{k+1} + \sum_{i=1}^{n} \mathbf{K}_i\boldsymbol{\rho}_{i,k+1} - \mathbf{L}^T\,\mathbf{q}_{k+1} = \mathbf{0}$$

$$(\boldsymbol{\delta}_{k+1} - (1+\gamma_i\Delta t\beta)\boldsymbol{\rho}_{i,k+1}) - (\boldsymbol{\delta}_k - (1-\gamma_i\alpha\Delta t)\boldsymbol{\rho}_{i,k}) = \mathbf{0}$$

$$(\mathbf{L}\boldsymbol{\delta}_{k+1} + \beta\Delta t\boldsymbol{\Phi}\mathbf{q}_{k+1}) + (-\mathbf{L}\boldsymbol{\delta}_k + \alpha\Delta t\boldsymbol{\Phi}\mathbf{q}_k) = 0.$$

If $\boldsymbol{\delta}_k$, $\boldsymbol{\delta}_{k+1}$ are eliminated from these equations they may be written

$$(\mathbf{A} + \beta\Delta t\mathbf{1})\xi_{k+1} = (\mathbf{A} - \alpha\Delta t\mathbf{1})\xi_k \tag{A1}$$

where $\alpha = 1-\beta$

$$\mathbf{A} = \begin{bmatrix} (1+\mathbf{K}_0^{-1}\mathbf{K}_1)/\gamma_1, & \mathbf{K}_0^{-1}\mathbf{K}_2/\gamma_1, & \dots, & \mathbf{K}_0^{-1}\mathbf{K}_n/\gamma_1, & -\mathbf{K}_0^{-1}\mathbf{L}^T/\gamma_1 \\ \mathbf{K}_0^{-1}\mathbf{K}_1/\gamma_2, & (1+\mathbf{K}_0^{-1}\mathbf{K}_2)/\gamma_2, & \dots, & \mathbf{K}_0^{-1}\mathbf{K}_n/\gamma_2, & -\mathbf{K}_0^{-1}\mathbf{L}^T/\gamma_2 \\ & & , & (1+\mathbf{K}_0^{-1}\mathbf{K}_n)/\gamma_n, & -\mathbf{K}_0^{-1}\mathbf{L}^T/\gamma_n \\ -\boldsymbol{\Phi}^{-1}\mathbf{L}\mathbf{K}_0^{-1}\mathbf{K}_1, & -\boldsymbol{\Phi}^{-1}\mathbf{L}\mathbf{K}_0^{-1}\mathbf{K}_2, & \dots, & -\boldsymbol{\Phi}^{-1}\mathbf{L}\mathbf{K}_0^{-1}\mathbf{K}_n, & \boldsymbol{\Phi}^{-1}\mathbf{L}\mathbf{K}_0^{-1}\mathbf{L}^T \end{bmatrix}$$

and $\boldsymbol{\xi}^T = (\boldsymbol{\rho}_1{}^T, \boldsymbol{\rho}_2{}^T, \dots, \boldsymbol{\rho}_n{}^T, \mathbf{q})$.

Now suppose the matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ with corresponding eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \dots, p_N$ then as is well known

$$\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1}$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$$

$$\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N].$$

It therefore follows that

$$\boldsymbol{\eta}_{k+1} = \mathbf{M}\boldsymbol{\eta}_K$$

where

$$\boldsymbol{\eta} = \mathbf{P}^{-1}\boldsymbol{\xi}$$

and

$$\mathbf{M} = \operatorname{diag}\left(\frac{\lambda_1 - \alpha\Delta t}{\lambda_1 + \beta\Delta t}, \ldots, \frac{\lambda_N - \alpha\Delta t}{\lambda_N + \beta\Delta t}\right).$$

So that

$$\boldsymbol{\eta}_{k+1} = \mathbf{M}^{k-1}\boldsymbol{\eta}_0.$$

It now becomes evident that the iterative solution will only remain stable provided.

$$\left|\frac{\lambda_i - \alpha\Delta t}{\lambda_i + \beta\Delta t}\right| \leq 1 \qquad \text{for all } i = 1, \ldots, N. \tag{A2}$$

Now the eigenvalues λ_i are all real and positive to show this observe that

$$(\mathbf{A} - \lambda\mathbf{1})\mathbf{p} = \mathbf{0}$$

implies that there exist nontrivial $\boldsymbol{\delta}$, $\mathbf{q}$ such that

$$\bar{\mathbf{K}}(s)\boldsymbol{\delta} - \mathbf{L}^T\mathbf{q} = \mathbf{0} \tag{A3}$$

$$-\mathbf{L}\boldsymbol{\delta} - \frac{1}{s}\boldsymbol{\Phi}\mathbf{q} = 0$$

where

$$\boldsymbol{\delta} = [\mathbf{K}_0^{-1}\mathbf{K}_1, \mathbf{K}_0^{-1}\mathbf{K}_2, \ldots, \mathbf{K}_0^{-1}\mathbf{K}_n, -\mathbf{L}^T]\mathbf{p} \tag{A4}$$

$$s = -1/\lambda$$

and $\bar{K}(s)$ is defined by eqn (11a).

First suppose that s is real then eqns (A3) and (A4) imply that

$$\boldsymbol{\delta}^T\bar{\mathbf{K}}(s)\boldsymbol{\delta} + \frac{1}{s}\mathbf{q}^T\boldsymbol{\Phi}\mathbf{q} = 0$$

that is

$$\boldsymbol{\delta}^T\left[\int_V \mathbf{B}^T\left\{\mathbf{D}_0 + \sum_{i=1}^{n}\frac{s}{s+\gamma_i}\mathbf{D}_i\right\}\mathbf{B}\,\mathrm{d}V\right]\boldsymbol{\delta} + \mathbf{q}^T\left[\int_V (\mathbf{E}^T\mathbf{k}\mathbf{E})\,\mathrm{d}V\right]\mathbf{q} = 0. \tag{A5}$$

Now suppose λ is negative so that $s = -1/\lambda$ is positive, then clearly since $\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n, \mathbf{k}$ are all positive definite eqn (A5) is the sum of positive definite terms, this of course implies that $\boldsymbol{\delta} = 0, \mathbf{q} = 0$ a contradiction and thus if λ is real λ must be positive.

Next suppose that λ is a complex eigenvalue it then follows from eqn (A1) that its complex conjugate λ^* is also an eigenvalue and thus that

$$\boldsymbol{\delta}^T\bar{\mathbf{K}}(s)\boldsymbol{\delta} + \frac{1}{s}\mathbf{q}^T\boldsymbol{\Phi}\mathbf{q} = 0$$

$$\boldsymbol{\delta}^T\bar{\mathbf{K}}(s^*)\boldsymbol{\delta} + \frac{1}{s^*}\mathbf{q}^T\boldsymbol{\Phi}\mathbf{q} = 0.$$

Subtracting these two equations and dividing by $s^* - s$ it is found that

$$\boldsymbol{\delta}^T\left[\int_V \mathbf{B}^T\left(\frac{\mathbf{D}_1}{|s+\gamma_1|^2} + \cdots + \frac{\mathbf{D}_n}{|s+\gamma_n|^2}\right)\mathbf{B}\,\mathrm{d}V\right]\boldsymbol{\delta} + \left|\frac{1}{s}\right|^2\mathbf{q}^T\left[\int_V (\mathbf{E}^T\mathbf{k}\mathbf{E})\,\mathrm{d}V\right]\mathbf{q} = 0.$$

Now because of the assumed positive definiteness of $\mathbf{D}_1, \ldots, \mathbf{D}_n, \mathbf{k}$ the above equations consist of the sum of positive definite terms and so implies that $\boldsymbol{\delta} = \mathbf{0}$ $\mathbf{q} = \mathbf{0}$ a contradiction.

Since the eigenvalues λ_i are all positive eqn (A2) implies that the integration process is always stable provided

$$\Delta t \leq \frac{\lambda_i}{(1/2) - \beta} \qquad 0 < \beta < \frac{1}{2}$$

$$\Delta t \geq \frac{\lambda_i}{(1/2) - \beta} \qquad \frac{1}{2} \leq \beta \leq 1.$$

Thus the process will always be stable when $\beta \geq 1/2$ but is only conditionally stable when $\beta < 1/2$.